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Applications of First-Order Ordinary Differential Equations to Real-Life Systems

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Abstract: *This article discussed applications of first-order ordinary differential equations to real-life systems, various types of differential equations with examples are presented.*

When a dead body is discovered somewhere, Police and detective security agencies are eager to identify the time of death and what caused the death. In this article, Newton's Law of Cooling is used to estimate the time of death of a dead body discovered at midnight with the aid of information about the body's surrounding temperature. It was discovered in this study that Newton's law of cooling only works when the temperature of the body's surroundings is kept constant.

The study also considered other applications of first-order differential equations such as the population growth model and radioactive decay of radioactive isotopes and illustrative examples are given in each case.

INTRODUCTION

Many physical phenomena in the field of science, engineering and technology when developed mathematically yield differential equations. This set of equations is called a model. The most important aspect of applied mathematics that has gained striking attention is the formulation of mathematical models using a set of differential equations to address real-life phenomena [1] some useful applications of differential equations are found in the areas of population growth and decay, distribution of the drug in the human body, carbon dating, wave in composite media, aerodynamics, casting of materials, electromagnetics analysis for detection of bar radar, rocket launch trajectory analysis, motion of a space vehicle, heat transfer and temperature problems which apply Newton's law of cooling [5,6]. A differential equation is an equation involving the derivatives of independent variables concerning one or more dependent variables. When it contains a single dependent variable it is called an ordinary differential equation which is our major focus in this study. It is known as a partial differential equation if it involves several independent variables and one or more dependent variables. Frigon and Pouso [2] studied the theory and applications of first-order ordinary differential equations which transformed the usual derivatives by Stielties derivatives. Rahan [4] investigated the first-order differential equation and Newton's law of cooling. Some relevant works in the field of differential equations are found in [3, 7, 8]. TarRan[9] also studied applications of first-order ordinary differential equations in engineering analysis.

2. Preliminaries

2.1 Basic Steps in Building a Differential Equation that Describes Real Life System (Model)

The procedures below help to build a model.

1. Identify the real-life problem correctly
2. State the mathematical assumptions on which the model will be developed, considering the variables
3. Describe fully the variables and parameters of the model
4. Apply the assumptions (in step 2) to formulate differential equations considering the parameters and variables described (in step 3)
5. Solve the formulated model (in step 4) and carry out real-life interpretation of the model for further application.

2.2 Type of Differential Equations with Examples

Ordinary Differential Equation (ODE):

Given a function *f an x, derivatives* general fn ordinary differential equation is written as $f(x, y, y', y'', ., ., ., y^{(n)}) = 0$

Examples of ODEs: $\frac{dy}{dx} = 5tanx$, $\frac{dy}{dx} = x^3$

Partial Differential Equation (PDE):

A partial differential equation is an equation involving an unknown function of two or more variables and its partial derivative concerning the variables [10] PDEs: $\frac{\partial^2 z}{\partial x^2 y} = e^y sinx$, $\frac{\partial^2 z}{\partial x^2 y} - 2xy^2 = 0$

2.3 Order and Degree of a Differential Equation.

The order of a differential equation is the highest derivative present in the equation whereas the degree of a differential equation is the power to which the highest derivative is raised.

For example, the differential equation

$the\ \frac{\partial^2 y}{\partial x^2} = \left(\frac{dy}{dx}\right)^4 + 2xy = 0$ is of order 2 and degree 1, whereas the differential equation

$the\ \left(\frac{\partial^2 y}{\partial x^2}\right)^4 + \left(\frac{dy}{dx}\right)^5 + 2xy = 0$ is of order 2 and degree 4

2.4. Linear and Non-linear Differential Equation.

A linear differential equation can be expressed in the form $y' + p(x)y = q(x)$

whereas a non-linear differential equation is not a linear equation in the unknown function and its derivative which cannot be expressed in the form $y' + p(x)y = q(x)$ but can be expressed as $y' + p(x)y = \alpha(x)y$.

3.0. Some Applications of First Order Differential Equation to Real Life Systems.

These are numerous real life applications of first-order differential equations to real life systems. In this study we shall discuss the following

- Population growth and decay
- Newton's law of cooling
- Radioactive decay

3.1. Population Growth and Decay

Population growth involves a dynamic process which can be developed using differential equations. The exponential growth model or natural growth model is known as Mathus' model [12]. This model is based on the assumption that the rate of change of the population is proportional to the existing population itself. If p(t) represents the total population at time t, the above assumption can be written as

$$\frac{dp}{dt} = kp(t) \quad (3.1)$$

Where k is the proportionality constant.

The above model (3.1) can also be used in the financial institute for example, when saving money in the bank, the balance in the savings account with interest compounded continuously exhibits natural growth provided no withdrawal and in this case the constant k represents the annual rate of interest, group of animal populations grows exponential provided size is not affected by environmental factors, in this case, k is known as the productivity rate of population and it can also be used in migration.

Tractor factory the equation wit e^{-kt}, the integrating factor$e^{-kt}\frac{dp}{dt} = kpe^{-kt}$

$e^{-kt}\frac{dp}{dt} - kpe^{-kt} = 0\ \frac{dp}{dt}[pe^{-kt}]= 0$

$\int \frac{dp}{dt}[pe^{-kt}] = \int 0$

$pe^{-kt} = C\ or\ p - ce^{kt}$

Suppose the initial population is p_0 then $p(0) = p_o\ and\ c = p_0$

$$p(t) = p_o e^{kt} \quad (3.2)$$

When k > 0 the population grows and when k < 0, the population decays

Example 1.

Suppose the population of a certain community is known to increase at a rate proportional to the number of people living in the community at time t, the population has doubled after 7 years, how long would it take to triple? If it is known that the population of the community is 12,000 after 5 years, determine the initial population and predict the population in 40 years.

Solution

Let p_0 denote the initial population of the community and $p(t)$ the population of the community at any time t, then from (3.1) we have

$\frac{dp}{dt} = kp \quad p(t) = p_0 e^{kt}$

From (3.2) given that

$p(7) = 2p_o \;\; e^{7k} = 2$

$k = \frac{0.6931}{7} = 0.0990$

The solution of the model becomes

$p(t) = p_o e^{0.0990t}$ (3.3)

Let t, be the time taken for the population to triple then $3p_0 = p_o e^{0.0990t} \;\; e^{0.0990t} = 3$

0.0990t = ln3

$t = \frac{1.0986}{0.0990} = 11.0970 \approx 11 years$

Applying $p(5)$ = 12,000

$12{,}000 = p_0 e^{0.0990 \times 5}$

$p_o = \frac{12{,}000}{e^{0{,}450}} = 7{,}315$

Hence the initial population of the community was

$p_o \approx 7{,}315$

Therefore, solution of the model is

$p(t) = 7315e^{0.0990t}$

So that the population in 40years is

$p(40) = 7315e^{40(0.0990)}$

$p(40) = 7315e^{3.960}$

$p(40) = 7315(52.4573)$

$p(40) \approx 383{,}725$

3.2. RADIOACTIVE DECAY

In physics and chemistry, a radioactive element disintegrates when it emits energy in form of ionizing radiation. Substances that emit ionizing radiation are known as radionuclides. When a radioactive substance decays, a radionuclide transforms into different atom-a decay product. The atoms keep transforming to new decay products until a state is reached and are no longer radioactive. The radioactive law states that the probability per unit of time that a radioactive substance will decay is a constant and independent of time, which means that the number of nuclei undergoing decay per unit of time is proportional to the total number of nuclei in the given substance [11].

The mathematical expression of the radioactive law is.$\frac{dA}{dt} \; \alpha \; A$

$\frac{dA}{dt} = kA$ (3.4)

Where A(t) is the amount of substance and k is the constant of proportionality.

Suppose the initial amount of the substance is A_o then

A(0) = A_o and solving (3.4) using the initial condition we have

$A(t) = A_o e^{kt}$ (3.5)

Equation (3.5) is the solution of (3.4) where the constant k can be obtained from the half-life of the radioactive substance.

The half-life of a radioactive material can be defined as the time it takes for one-half of the atom in an initial amount (A_o) to transform into atoms of the new element. Half-life determines the stability of a radioactive element. The half-life of a radioactive substance is directly proportional to its stability.

Let T be the half-life of a radioactive element, then caphen$A(T) = \frac{A_o}{2}$ (3.6)

Applying (3.5) and (3.6) we have,

$\frac{A_0}{2} = A_o e^{kt}$

$T = \frac{-ln}{k}$ (3.7)

Example 2

If the half-life of a radioactive element is 18days and we have 40g at the end of 30 days. Determine the amount of radioactive element present initially

Solution

Let A(t) represent the amount present at time t and Ao the initial amount of the element.

$\frac{dA}{dt} = kA$

A(40) = 30

Solving the IVP, yields.

$A(t) = A_0 e^{kt}$

But from (3.7)

$k = \frac{-ln2}{T}$

$k = \frac{-ln2}{18}$ (3.8)

Applying A(40) = 30

$40 = A_o e^{30k}$

$A_o = 40e^{-30k}$ (3.9)

Using (3.8) we have

$A_o = 40e^{\frac{30ln2}{18}}$

$A_o = 127g$

3.3 Newton's Law of Cooling

Another important real life application of differential equation is Newton's law of cooling. Sir Isaac Newton developed a huge interest in quantitative findings of the loss of heat in a body and a formula was derived to represent this phenomenon. The law states that the rate of change of temperature of a body is directly proportional to the difference between is solid object and the surrounding environment at a given instant of time.$\frac{dT}{dt} \alpha (T_0 - T_5)$

$\frac{dT}{dt} = k(T_0 - T_5)$

(3.10)

Where T(0) = T_0

$T_0 =$ Temperature of the body

$T_5 =$ Temperature of the surrounding

K = Constant of proportionality

$\int \frac{dT}{T_0 - T_5} = \int kdt$

$ln|T_0 - T_5| = kt + c$

$T_0 - T_5 = e^{kt+c}$

Applying T(0) = T_0 yields

$C = T_0 - T_5$

$T(t) = T_5 + (T_0 - T_5)e^{kt}$

Suppose the temperatures at t_1 and t_2 are given we have

$T(t_1) - T_5 = (T_0 - T_5)e^{kt_1}$

$T(t_2) - T_5 = (T_0 - T_5)e^{kt_2}$

So that

$\frac{T(t_1) - T_5}{T(t_1) - T_5} = e^{k(t_1 - t_2)}$ (3.11)

Example 3

A police man discovered a dead body at midnight in a room where the temperature of the dead was 90°F, the body temperature of the room was kept constant at 70°F. Three hours later the temperature of the body dropped to 85°F. Determine the time of death of the victim

Solution $T(0) = 98.6^oF$ $(37^0c) = T_o$ and $T_5 = 70^oF$

Provided the victim was not sick

$\frac{dT}{dt} = k(T_0 - 70),\ T(0) = 98.6$

But

$T(t) = T_5 + (T_0 - T_5)e^{kt}$

So that

$\frac{T(t_1) - T_5}{T(t_2) - T_5} = e^{k(t_1 - t_2)}$

$T(t_1) = 90^oF \ and \ T(t_2) = 85^oF$

$\frac{90-70}{85-70} = e^{3k}$

$t_1 - t_2 = 3hours$

$k = \frac{1}{3} ln \frac{4}{3} = 0.0959$

Let t_1 and t_2 2 represent times of death and discovery of the dead body then

$T(t_1) = T(0) = 98.6^oF$

And $T(t_2) = 90^oF$

The time of death $(t_3) = t_1 - t_2$ and from (3.11) we have

$$\frac{T(t_1)-T_5}{T(t_2)-T_5} = e^{kt_3} \quad (3.12)$$

$\frac{98.6-70}{90-70} = e^{kt_3}$

$t_3 = \frac{1}{k} ln \frac{28.6}{20} = \frac{1}{0.0959} ln \frac{28.6}{20} \approx 3.730$

Therefore the person died at about 8:18 pm

4. Conclusion

Many physical problems in the fields of science, economics, engineering and technology remain meaningless without the application of differential equations to transform them into models. In this article real life problems are solved with the aid of differential equations. our concentration is on population growth and decay, radioactive decay and Newton's law of cooling. The physical growth and decay of any population which is well discussed in this article is of great concern to humanity. We used the population growth model to predict the population of a given community in 40years' time, this means that the population growth model can be used to predict the population of a country in future when some facts about the country are known. This can help the government of such a country to plan ahead of time and equally embark on population control measures when necessary. Radioactive decay is of great importance for the nucleus as the decay transforms it into a stable state, many of these modern technologies are products of radioactive decay, and a large amount of energy can be generated using decay in nuclear rector which is then converted to electrical energy for use in various form, in medical science, radioactive isotopes which can undergo radioactive has a great application because these isotopes are referred to as tracers and are injected into the body of a patient, in the body, the tracers gives off harmless radiation though may be detected through the device and this detection, scientists (physicians) can investigate blood flow to specific organs and evaluate organ function or bone growth [11]. There are numerous such applications of differential equations such as drug distribution in the human body, survivability with HIV/AIDS, underwater acoustic signal processing, crystal growth, transportation and distribution of chemicals through the body, radio interferometry, seismic wave propagation in the earth (earthquake), aircraft landing field length. It article showed that differential equations have numerous applications in real life systems.

Conflict of Interests

The authors declare that they have no conflict of interest.

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